An Intransitive Model of the Earth-Atmosphere-Ocean System

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ABSTRACT

The earth is regarded as a whole system. By considering the sources, sinks and flows of energy, a simple steady-state model is developed. It is found that there are five solutions, corresponding to five climates, consistent with present values of the model's parameters. One of these corresponds quite accurately to the present climate of the earth. Another corresponds to what might be necessary for an ice age. When perturbed slightly, several of the solutions (including the one corresponding to the earth's present climate) are found to become unstable. The only completely stable climate is one for which the earth is ice-covered.

1. Introduction

The number of global climatic models which have been developed and solved by the scientific community to date has been small. Those which are capable of being solved completely (e.g., Budyko, 1969; Sellers, 1969) are necessarily quite simple. Although they differ in various aspects, their results, in general, have been quite similar (Budyko, 1970; Sellers, 1970). Specifically, in Budyko's model when the radiation from the sun is decreased by 1.6%, the ice cover reaches the mean latitude of about 50°. Further reduction in radiation starts the ice shifting toward lower latitudes and finally to the equator as a result of self-development. Sellers' model has similar results: a 2% decrease in radiation causes the icecaps to extend to 50°, and a further decrease results in a rapid transition to an ice-covered earth.

This paper summarizes a thesis (Faegre, 1971) in which a simple global climatic model, similar to those of Budyko and Sellers, is developed, solved and perturbed. What is interesting is that the results are similar in interpretation to those of Budyko and Sellers, but fundamentally different in form.

The model of this paper results in several solutions which correspond to alternative, physically possible climates. Lorenz (1970) terms this an intransitive model, as opposed to a transitive model, for which there is only one solution. Both Budyko's and Sellers' models are examples of the latter type.

2. The model

The approach used to develop the model was essentially that of Sellers (1969), but simpler in some respects. If one is worried about the problem of oversimplification (as one should be), this model can alternatively be viewed as a more complex version of one done by Fritz (1960). The energy balance equation for the time-averaged, one-dimensional, earth-atmosphere-ocean system is written:

$$\frac{1}{r\cos\phi}\frac{d}{d\phi}(\rho v\cos\phi)=S_0+S_i, \tag{1}$$

where ρv is the energy flow, S_0 are the sources, S_i the sinks, ϕ the latitude, and r the earth's radius.

The energy flow, ρv, is considered as the sum of three flows: sensible heat c in the atmosphere, latent heat C in that atmosphere, and sensible heat F in the ocean; thus,

$$\rho v=c+C+F. \tag{2}$$

These flows are considered as diffusion-type phenomena and are thus mathematically expressed as diffusion equations, i.e.,

$$c=-\frac{c_p}{g}p_0\alpha_A\frac{1}{r}\frac{dT}{d\phi}, \tag{3}$$

$$C=-\frac{L_{wv}p_0}{g}\alpha_w\frac{1}{r}\frac{dq}{d\phi}, \tag{4}$$

$$F=-z_0\gamma c_w\alpha_0\frac{1}{r}\frac{dT}{d\phi}, \tag{5}$$

where the notation is explained in the Appendix.

Using the Clausius-Clapeyron equation, Eq. (4) may be expressed as a function of temperature such that

$$C=-\frac{(L_{wv})^2\epsilon e_{s0}}{gR_vT^2}\exp\left[\frac{L_{wv}}{R_v}\left(\frac{1}{273}-\frac{1}{T}\right)\right]\alpha_w\frac{1}{r}\frac{dT}{d\phi}. \tag{6}$$

The sun is considered as the only energy source and is expressed as

$$S_0 = Q_s(1-A), \tag{7}$$

where Q_s is the incident solar radiation and A the earth's albedo. Following Sellers' approach, data on the earth's albedo and temperature are used to find the following simple linear formula to relate the two:

$$A = 0.4860 - 0.0092(T-273). \tag{8}$$

Upper and lower limits of 0.85 and 0.25 are used to keep the albedo within physically realizable situations.

Infrared emission to space is considered the only sink and is expressed as

$$S_i = -K\sigma T^4, \tag{9}$$

where σ is the Stefan-Boltzman constant and K an empirical function of latitude adjusted to fit existing conditions.

Sellers has additional terms in each of Eqs. (3), (4) and (6) that represent transport through meridional circulations. He empirically finds such a term to be proportional to the difference between the temperature gradient at that latitude and the averaged absolute value of the temperature gradient over the whole globe. For Eq. (9) Sellers used the empirically determined formula

$$S_i = -\sigma T^4[1 - m \tanh(19T^6 \times 10^{-16})],$$

where m is a constant. These differences are presumably major constraints that cause the substantial difference in results between Sellers' model and this model.

Values of the parameters K, p_0, α_0, α_A and α_w are determined from data given in Sellers' paper. For specific values used the reader is referred to Faegre (1971, pp. 80–82). The form of the model's results is not altered by substantially varying the values of these parameters. The function γ is determined from an atlas and Q_s from any standard climatology text.

Eqs. (2), (3), (5), (6), (7), (8) and (9) are substituted into Eq. (1), resulting in a second-order, nonlinear differential equation. It is solved by conversion to finite-difference form and integration on an IBM 1130 computer with the boundary conditions that there is no energy flow off either pole.

The actual process is one of guessing an initial temperature at the north pole, assuming the boundary condition is met at that pole, and then successively calculating the temperature at 5° latitude intervals until the south pole is reached at which time it can be calculated how closely the boundary condition is met at that pole. To find solutions, the computer successively tries temperatures for the north pole from 100 to 300K at 5K intervals. From the computer print-out of how closely the southern boundary condition is met for a given temperature at the north pole,

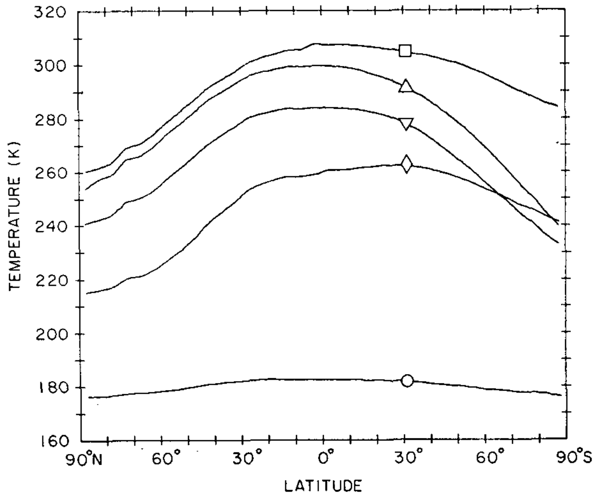

FIG. 1. The five climate solutions of the model: △ represents the earth's present climate, ▽ a climate that might correspond to that of an ice age, and ○ an ice-covered earth. The climates corresponding to □ and ◇ are asymmetric and difficult to account for physically.

it is then possible to see where the solutions are and to find them to a certain accuracy. A certain amount of time was spent looking for solutions with initial northern boundary temperatures below 100K and above 300K but with no success.

3. Results

The most interesting result is the finding that there is not just one temperature distribution consistent with Eq. (1), but five. Conveniently, one of the distributions corresponds to the present climate of the earth. We should not be too surprised by this result, because empirical data are used to find many of the parameters of the model.

Fig. 1 gives the temperature distributions of the five different climates consistent with Eq. (1). The temperature-latitude curve (climate) labeled △ corresponds to our present climate. The climate labeled ○ corresponds to an ice-covered earth. The climate labeled ▽ represents a climate very similar to our present one except that it is 15C colder on the average. The climates labeled □ and ◇ are difficult to account for physically. They correspond to very asymmetric solutions where temperatures of the Southern Hemisphere are considerably higher than those of the Northern. In discussing the five climates, the above symbols will be their names.

It is found that if the parameter K of Eq. (9) is given a constant value (the averaged value weighted with respect to latitudinal circumference), the two asymmetric solutions □ and ◇ disappear from the solution set. The assumption is made that they are just spurious solutions caused by the inexactness of the original radiation formula.

Each of the climates is perturbed by a ±1%, ±2%, and +4% increase in radiation. This allows a simple test of the climates' sensitivities to changes in the intensity of the sun's radiation. In general, decreasing radiation results in decreasing temperatures and vice versa. However, it is found that changes in radiation on a large enough scale often result in particular solutions becoming unstable. This implies that for these solutions the steady-state condition no longer satisfies the imposed boundary conditions.

Thus, for the present climate △, a decrease in radiation of 2% or an increase in radiation of 4% results in instability. For the climate ▽, just below our present climate, a decrease in radiation of 2% results in instability. The ice-covered earth, the climate labeled ○, is stable for all perturbations tried.

4. Conclusions

A pertinent indication of the model is that the present climate △ does not appear to be very stable. Both climates △ and ▽ are unstable for a radiation decrease of 2%, and climate ○ is stable for any perturbations. We conclude that for a 2% decrease in radiation the only solution is the climate ○, the ice-covered earth. Thus, the results of this model are similar in interpretation to those of Budyko and Sellers. However, the other models are transitive: there is one solution which transforms continuously as it is perturbed. In contrast, the model of this paper, with its multiple solutions, is clearly intransitive.

A basic consequence of this difference is as follows. If the earth is perturbed into an ice-covered climate, Budyko's and Sellers' models predict a return to the normal climate when the radiation returns to its original value. In the model of this paper, if the earth becomes ice-covered it remains ice-covered, even when radiation is returned to the unperturbed value. Thus, such a change would be irreversible.

If temperatures <-10C are considered as the necessary conditions for an ice cap, we see that the climate ▽ has ice-caps extending to ~50° latitude. It is tempting to consider climate ▽ as the one necessary for an ice age and to hypothesize the whole earth system as oscillating back and forth between the present climate △ and the climate ▽.

The simplicity of the model being what it is, it will be interesting to see what more sophisticated models predict with respect to these phenomena.

APPENDIX

Notation

Variables

- q specific humidity
- ϕ latitude
- T temperature (°K unless noted)
- A albedo
- Q_s incident radiation from sun
- α_0 constant of proportionality for oceanic transport
- α_w constant of proportionality for water vapor transport
- α_A constant of proportionality for atmospheric transport
- γ the fraction of a latitude circle that is ocean
- p_0 sea level atmospheric pressure
- K empirical correction for black body radiation

Constants

- σ Stefan-Boltzman constant [1.356×10^{-12} cal $cm^{-2}(°K)^{-4}sec^{-1}$]
- z_0 average depth of ocean that transports heat (6×10^3 cm)
- c_w heat capacity of water [1 cal gm^{-1} $(°K)^{-1}$]
- r radius of earth (6.37×10^8 cm)
- L_{wv} latent heat of evaporation (590 cal gm^{-1})
- g force of gravity (9.8×10^2 cm sec^{-2})
- c_p specific heat of dry air at constant pressure [0.24 cal gm^{-1} $(°K)^{-1}$]
- e_{s0} saturation vapor pressure at 273K (6.1×10^4 dyn cm^{-2})
- ϵ ratio of molecular wt of water vapor to that of dry air (0.622)
- R_v gas constant for water vapor [0.11 cal gm^{-1} $(°K)^{-1}$]

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